# An Introduction to Optimization Techniques in Computer Graphics

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# Variational Methods

## Continuous convex models and optimization

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# Overview

1. Introduction to variational methods
2. Convex optimization primer: proximal gradient methods
3. Variational inverse problems
4. Segmentation and labeling problems
5. 3D reconstruction
6. Summary

# Overview

1. **Introduction to variational methods**
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# Fundamental problems in computer vision

## Image labeling problems

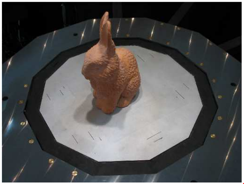
Segmentation
and Classification

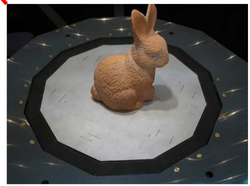
Stereo

Optic flow

# Fundamental problems in computer vision

3D Reconstruction





- **Unknown object**: **Vector-valued function**

$$u : \Omega \to \mathbb{R}^m$$

on a **continuous** domain $\Omega$.

- **Problem solution**: minimizer of an **energy functional**

$$\underset{u \in \mathcal{V}}{\operatorname{argmin}} \Big\{ \underbrace{J(u)}_{\text{regularizer}} + \underbrace{F(u)}_{\text{data term}} \Big\}.$$

on an infinite dimensional (function) space $\mathcal{V}$.

# Continuous vs. discrete methods

**Advantages:**

- Realistic physical models of the world naturally continous
- Exact modeling of geometric quantities (length, curvature) ...
- No grid bias (rotation invariance)
- Solvers easy to parallelize, fast on GPU

**Disadvantages:**

- Less flexible than probabilistic graphical models
- Iterative solvers (usually no runtime guarantee)
- Slow on single core

# Classical variational model: total variation denoising

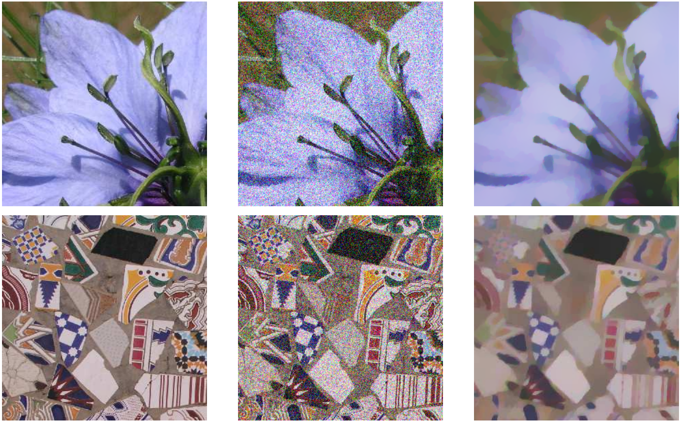
**Rudin-Osher-Fatemi (ROF), or TV-$\mathcal{L}^2$ model (1992)**

Given an image $f$ and a smoothness parameter $\lambda > 0$, compute a **denoised** image as the minimizer

$$\underset{u \in L^2(\Omega)}{\operatorname{argmin}} \left\{ \|\nabla u\|_1 + \frac{1}{2\lambda} \|u - f\|_2^2 \right\}.$$

- *Can be interpreted as maximum a posteriori (MAP) estimate under assumption of Gaussian noise on $f$.*
- *The energy is convex, but not differentiable,*
  *in particular gradient based methods do not work*

# TV-$\mathcal{L}^2$ denoising examples



*Original* *Noisy* *Solution*

# Solving the TV-$\mathcal{L}^2$ model

- The TV-$\mathcal{L}^2$ model

$$\underset{u}{\operatorname{argmin}} \left\{ \|\nabla u\|_1 + \frac{1}{2\lambda} \|u - f\|_2^2 \right\}.$$

  is a building block for several more general algorithms. It is therefore important to fully understand it.
- Note that the total variation is not differentiable, so it cannot be exactly minimized by simple gradient descent, although it is convex.
- In this section, we discuss discretization and the most straight-forward exact solver I know of.

# Grayscale images

In the continuous setting, an image is a function or scalar field $u : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^2$. For simplicity, we assume in this course that the domain is always two-dimensional.

### Discretization of an image

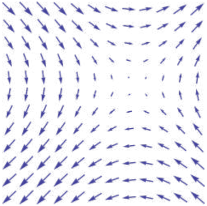
- the domain $\Omega$ is a regular grid of size $M \times N$
- an image $u$ is a matrix $u \in \mathbb{R}^{M \times N}$
- the gray value or intensity at a pixel $(i, j)$ is written as $u_{i,j}$

# Color images

**Color images** have several components, usually red, green and blue for the primary colors. Thus, they are functions $u : \Omega \to \mathbb{R}^3$. In the discretization, each component $u^1$ is represented by its own matrix of intensity values.

# Vector fields

A **vector field** is a map $\mathbf{p} : \Omega \to \mathbb{R}^2$. One can imagine it as an assignment of a little arrow to each point:



### Discretization of a vector field

- a vector field $\mathbf{p}$ consists of two components $p^1, p^2$, each of which is a matrix of size $M \times N$
- the vector at a pixel $(i, j)$ is written as $\begin{bmatrix} p^1_{i,j} \\ p^2_{i,j} \end{bmatrix}$.

## The gradient of a scalar field

In the continuous setting, one of the basic operators acting on a function is the derivative. The vector field of the two partial derivatives of a function $u : \Omega \to \mathbb{R}$ is called the **gradient** of $u$,

$$\nabla u = \begin{bmatrix} \partial_x u \\ \partial_y u \end{bmatrix}.$$

### Discretization of the gradient with forward differences

For $i < N$ and $j < M$:

$$(\nabla u)_{i,j} = \begin{bmatrix} u_{i+1,j} - u_{i,j} \\ u_{i,j+1} - u_{i,j} \end{bmatrix}.$$

Neumann boundary conditions $\frac{\partial u}{\partial \mathbf{n}} = 0$ are used for functions, i.e.

$$(\nabla u)^1_{M,j} = 0 \text{ and } (\nabla u)^2_{i,N} = 0.$$

## The divergence of a vector field

In the continuous setting, the divergence of a vector field $\mathbf{p}$ is the scalar field or function $\mathrm{div}(\mathbf{p}) : \Omega \to \mathbb{R}$ defined as

$$\mathrm{div}(\mathbf{p}) = \partial_x p^1 + \partial_y p^2 .$$

**Discretization of the divergence with backward differences**

Derivatives on vector fields use Dirichlet boundary conditions $\mathbf{p}|_{\partial\Omega} = 0$. Thus,

$$(\mathrm{div}(\mathbf{p}))_{i,j} = \begin{cases} p^1_{i,j} - p^1_{i-1,j} & \text{if } 1 < i \leq M \\ p^1_{i,j} & \text{if } i = 1 \end{cases}$$
$$+ \begin{cases} p^2_{i,j} - p^2_{i,j-1} & \text{if } 1 < j \leq N \\ p^2_{i,j} & \text{if } j = 1 \end{cases}$$

# Discrete total variation

Let $u \in \mathbb{R}^{M \times N}$ be a matrix, then its **total variation** is defined as

$$\mathrm{TV}(u) := \sum_{i=1}^{M} \sum_{j=1}^{N} \left| (\nabla u)_{i,j} \right|_2 ,$$

where the gradient is computed with forward differences and Neumann boundary conditions.

**Note:** Actually computing the discrete TV requires to compute the sum over all elements in a matrix, which is a not fully parallelizable type of operation called a **reduction**.

# Dual formulation

The Euclidean norm in $\mathbb{R}^n$ can be written for any vector $v$ as

$$|v|_2 = \sup_{p \in \mathbb{R}^n, |p|_2 \leq 1} (v, p),$$

where $(v, p)$ denotes the usual inner product.

We can use this fact to rewrite the total variation of a matrix as follows:

$$\mathrm{TV}(u) = \max_{\mathbf{p} \in K} \sum_{i,j} \left( (\nabla u)_{i,j} \, , \, \mathbf{p}_{i,j} \right) ,$$

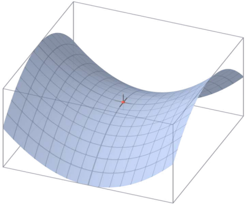
where the (convex) set $K$ consists of all vector fields such that their vectors in each point have length at most one,

$$K := \left\{ \mathbf{p} \in \mathbb{R}^{M \times N} \times \mathbb{R}^{M \times N} \ : \ \left| \mathbf{p}_{i,j} \right|_2 \leq 1 \text{ for all } i, j \right\} .$$

# Discrete saddle point formulation of TV-$L^2$

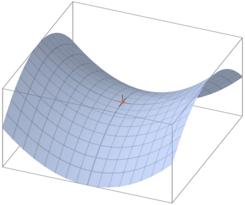
Instead of finding a minimizer for ROF, we can look for a **saddle point** $(u, \mathbf{p})$ of

$$\min_{u \in \mathbb{R}^{M \times N}} \max_{\mathbf{p} \in K} \sum_{i,j} \left( (\nabla u)_{i,j} \, , \, \mathbf{p}_{i,j} \right) + \frac{1}{2\lambda} (u_{i,j} - f_{i,j})^2 .$$



- Note that the energy is now differentiable, so as a simple method to find a saddle point we can iterate gradient descent in $u$ and ascent in $\mathbf{p}$ starting from an initial point $(u_0, \mathbf{p}_0)$.
- We will formulate a slightly faster method, which is based on the observation that for given $\mathbf{p}$, one can solve the minimization in $u$ explicitly.

## Gradient ascent in the dual variable



The energy of the saddle point problem is

$$E(u, \mathbf{p}) = \sum_{i,j} \left( (\nabla u)_{i,j} \, , \, \mathbf{p}_{i,j} \right) + \frac{1}{2\lambda}(u_{i,j} - f_{i,j})^2.$$

Its derivative with respect to **p** is just

$$\frac{d}{d\mathbf{p}} E(u, \mathbf{p}) = \nabla u,$$

so a (discrete) gradient ascent step for **p** is given by setting

$$\mathbf{p}^{n+1} = \mathbf{p}^n + \tau \nabla u^n,$$

where $(\mathbf{p}^n, u^n)$ is the $n$th iterate for the saddle point and $\tau > 0$ is a small enough step size (to be determined later).

# Projection onto $K$

The allowed range for $\mathbf{p}$ is the set $K$. After the update step, $\mathbf{p}^n$ might lie outside $K$, so we have to back-project it onto $K$. This can be done by projecting each single vector of the field $\mathbf{p}$ back onto the unit disk,

$$\Pi_K(\mathbf{p}^n)_{i,j} = \frac{\mathbf{p}^n_{i,j}}{\max\left(1, \left|\mathbf{p}^n_{i,j}\right|_2\right)}.$$

# Updating the primal variable

The reason the finite difference operators introduced in the first lecture were defined in this way is that they need to satisfy

$$\sum_{i,j} \left( (\nabla u)_{i,j} \, , \, \mathbf{p}_{i,j} \right) = - \sum_{i,j} u_{i,j} \, (\operatorname{div}(\mathbf{p}))_{i,j}$$

like their continuous counterparts.

This means we can rewrite the saddle point energy as

$$E(u, \mathbf{p}) = \sum_{i,j} -u_{i,j} \, (\operatorname{div}(\mathbf{p}))_{i,j} + \frac{1}{2\lambda} (u_{i,j} - f_{i,j})^2 .$$

## Condition for a saddle point and update step

The gradient with respect to $u$ is

$$\frac{d}{du}E(u, \mathbf{p}) = -\mathrm{div}(\mathbf{p}) + \frac{1}{\lambda}(u - f),$$

At a minimum for $u$, this must be zero. This leads to the update equation

$$u^{n+1} = f + \lambda \mathrm{div}(\mathbf{p}^{n+1})$$

which computes the exact solution for $u^{n+1}$ explicitly, given the previously computed $\mathbf{p}^{n+1}$.

# Complete algorithm

**Algorithm to solve ROF**
**(Bermudez-Moreno 1981, Chambolle 2005)**

- Given an image $f$ to be denoised and a smoothing parameter $\lambda > 0$.
- Set the step size to $\tau := \frac{1}{2\lambda\|\nabla\|} = \frac{1}{8\lambda}$.
- Start with initial $u^0$, $\mathbf{p}^0$, where $u^0$ is a scalar field and $\mathbf{p}^0$ is a vector field. Initial values can be arbitrary, choose e.g. zero for everything.
- Iterate until convergence:

$$
\begin{aligned}
\mathbf{p}^{n+1} &= \Pi_K(\mathbf{p}^n + \tau \nabla u^n) \\
u^{n+1} &= f + \lambda \operatorname{div}(\mathbf{p}^{n+1})
\end{aligned}
$$

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# Setting and notation

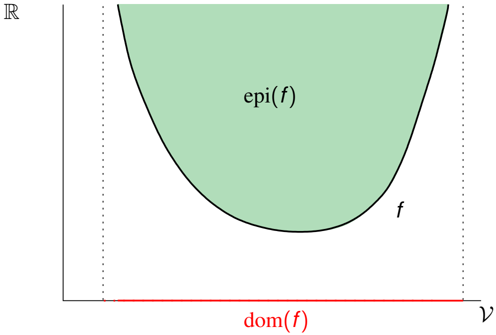
- A Hilbert space $\mathcal{V}$ with
  - inner product $(\cdot,\cdot)$ and
  - norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$,

  in this tutorial, this will be $\mathcal{V} = R^N$ or $\mathcal{V} = L^2(\Omega)$.
- The dual space $\mathcal{V}^*$ - in our setting, you can substitute $\mathcal{V}^* = \mathcal{V}$ if you are not familiar with the concept
- We consider "proper" functionals $f : \mathcal{V} \to \mathbb{R} \cup \{\infty\}$, i.e. they can take the value infinity, but we exclude the (boring) functional $f = \infty$ for technical reasons.

# The epigraph of a functional

The **epigraph** epi($f$) of a functional $f : \mathcal{V} \to \mathbb{R} \cup \{\infty\}$ is the set "above the graph", i.e.

$$\text{epi}(f) := \{(x, \mu) \ : \ x \in \mathcal{V} \text{ and } \mu \geq f(x)\}.$$



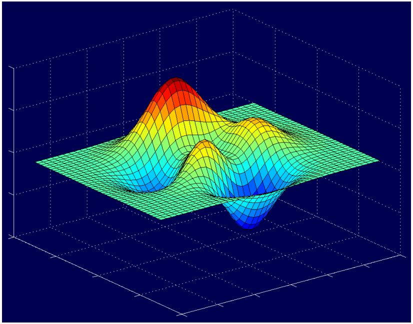
# Convex functionals

**Definition**

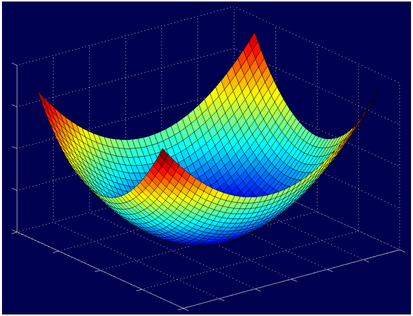
- A functional $f : \mathcal{V} \to \mathbb{R} \cup \{\infty\}$ is called convex if epi($f$) is a convex set.
- The set of all proper and convex functionals on $\mathcal{V}$ is denoted conv($\mathcal{V}$).

*We choose the geometric definition of a convex function here because it is more intuitive, the usual algebraic property is a simple consequence.*

# Convex versus non-convex energies



non-convex energy



convex energy

**Convex energies can be globally minimized -**
**for non-convex energies this is usually impossible.**

# The subdifferential

**Definition**

- A vector $\varphi \in \mathcal{V}^*$ is a **subgradient** of $f$ at $x \in \mathcal{V}$ if
$$f(y) \geq f(x) + \langle y - x, \varphi \rangle \text{ for all } y \in \mathcal{V}.$$
- The set of all subgradients of $f$ at $x$ is the **subdifferential** $\partial f(x)$.

*Geometrically: the function on the right hand side is an affine function*

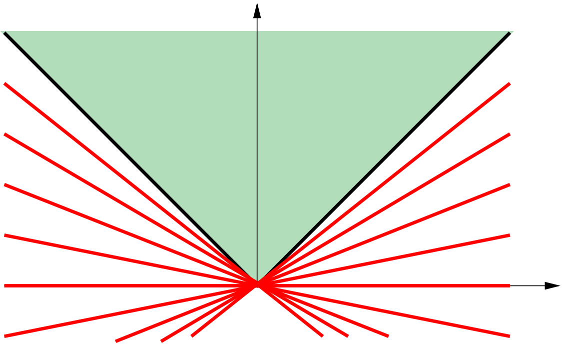
$$h(y) = f(x) + \langle y - x, \varphi \rangle$$

*with slope $\varphi$, which touches the epigraph of $f$ in $x$. The condition says that $h$ needs to lie below $f$.*

# The subdifferential

Example: the subdifferential of $f : x \mapsto |x|$ in 0 is

$$\partial f(0) = [-1, 1].$$



# Subdifferential and Fréchet derivative

*The subdifferential is a generalization of the Fréchet derivative (or the gradient in finite dimension), in the following sense.*

**Theorem**

Let $f \in \text{conv}(\mathcal{V})$ be Fréchet differentiable at $x \in \mathcal{V}$. Then

$$\partial f(x) = \{df(x)\}.$$

*The proof of the theorem is surprisingly involved - it requires to relate the subdifferential to one-sided directional derivatives. We will not explore this here, see e.g. [Rockafeller 1970].*

# Euler-Lagrange equation for convex functionals

**Theorem**

*Let $f \in conv(\mathcal{V})$. Then $\hat{x}$ is a* **global minimum** *of $f$ if and only if*

$$0 \in \partial f(\hat{x}).$$

*In particular, a convex functional does not have minima which are local but not global.*

*Geometrically intuitive, but the proof is also surprisingly simple. Rewriting the subgradient definition, one sees that $0 \in \partial f(\hat{x})$ just means that*

$$f(y) - \langle y, 0 \rangle \geq f(\hat{x}) - \langle \hat{x}, 0 \rangle \text{ for all } y \in \mathcal{V}.$$

*This already implies the equivalence.*

# Proximity operator

- The **proximity operator** for a closed convex functional $H$ and step size parameter $\tau > 0$ is defined as

$$\operatorname{prox}_{\tau H}(x) = \underset{y}{\operatorname{argmin}} \left\{ \frac{1}{2\tau} \|y - x\|^2 + H(y) \right\}.$$

- At the minimizer $\hat{x} = \operatorname{prox}_{\tau H}(x)$,

$$\begin{aligned} & 0 \in \frac{1}{\tau}(\hat{x} - x) + \partial H(\hat{x}) \\ \Leftrightarrow \quad & \hat{x} \in x - \tau \, \partial H(\hat{x}). \end{aligned}$$

The proximity operator computes an implicit subgradient descent for the convex functional $H$ with step size $\tau$.

# Examples for the prox operator (1)

- Let $H = TV$, then

$$\text{prox}_{\lambda TV}(x) = \underset{y}{\text{argmin}} \left\{ TV(y) + \frac{1}{2\tau} \|x - y\|^2 \right\}$$

is an instance of ROF-denoising.

- Solve e.g. with Bermudez-Moreno.

# Examples for the prox operator (2)

- Let $\mathcal{C} \subset \mathcal{V}$ be a convex set, and

$$\delta_{\mathcal{C}}(x) := \begin{cases} 0 & \text{if } x \in \mathcal{C}, \\ \infty & \text{otherwise} \end{cases}$$

  be its **indicator function**.

- Then

$$\begin{aligned} \operatorname{prox}_{\delta_{\mathcal{C}}}(x) &= \underset{y \in \mathcal{V}}{\operatorname{argmin}} \left\{ \frac{1}{2\tau} \|y - x\|^2 + \delta_{\mathcal{C}}(y) \right\} \\ &= \underset{y \in \mathcal{C}}{\operatorname{argmin}} \left\{ \|y - x\|^2 \right\} \end{aligned}$$

  is the projection onto $\mathcal{C}$.

- Adding an indicator function to a functional is equivalent to an optimization constraint.

# Examples for the prox operator (3)

- Let $H = \|\cdot\|_1$ (sparsity inducing norm), e.g. interesting in compressive sensing or dictionary learning, next talk.

- Then (component-wise)

$$\text{prox}_{\tau\|\cdot\|_1}(x)_i = (|x_i| - \tau)\text{sgn}(x_i),$$

which is called the **shrinkage operator**.

# Iterative shrinkage and thresholding (ISTA)

**Applicability:** Minimization problems of the form

$$\underset{u \in \mathcal{V}}{\operatorname{argmin}} \{G(u) + F(u)\}$$

such that

- $G$ is convex
- $F$ is convex and differentiable
- $dF$ is Lipschitz continuous, i.e. there exist $L > 0$ such that

$$\|dF(u) - dF(v)\| \leq L \|u - v\| \text{ for all } u, v \in \mathcal{V}.$$

- The proximation for $G$ is (comparatively) easy to evaluate.

# Iterative shrinkage and thresholding (ISTA)

**Algorithm:** Initialize with $u_0 = 0$,
then compute alternating "forward-backward steps":

- gradient descent in $F$:

$$v_{n+1} = u_n - \frac{1}{L} dF(u_n).$$

- implicit subgradient descent in $G$:

$$u_{n+1} = \text{prox}_{\frac{1}{L}G}(v_{n+1}).$$

**only $O(1/n)$ convergence (slow).**

Bruck 1977, Passty 1979

# Fast ISTA (FISTA)

**Algorithm:** Initialize with $u_0 = 0, \bar{u}_0 = 0, \mathbb{R} \ni t_0 = 1$, then iterate

- gradient descent in $F$:

$$v_{n+1} = \bar{u}_n - \frac{1}{L} dF(\bar{u}_n).$$

- implicit subgradient descent in $G$:

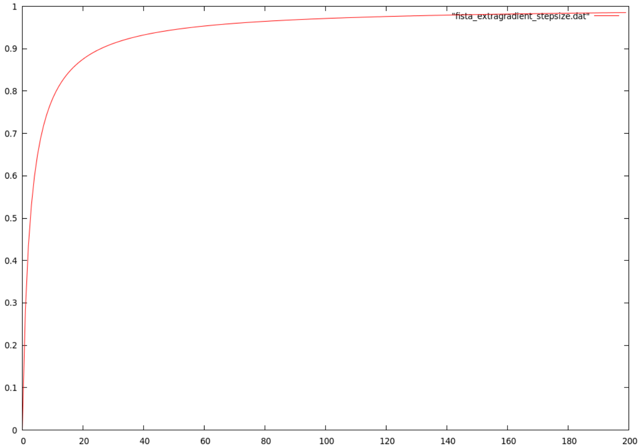
$$u_{n+1} = \operatorname{prox}_{\frac{1}{L}G}(v_{n+1}).$$

- extragradient step:

$$t_{n+1} = \frac{1}{2}\left(1 + \sqrt{1 + 4t_n^2}\right),$$
$$\bar{u}_{n+1} = u_n + \frac{t_n - 1}{t_{n+1}}(u_{n+1} - u_n).$$

Beck and Teboulle SIIMS 2009

# FISTA extragradient step size development




- Leads to $O(1/n^2)$ algorithm (optimal for a first-order method)
- Good general-purpose solver, easy to implement
  (in particular if proximation is already available)

## Example: constrained differentiable problems

Let $\mathcal{C} \subset \mathcal{V}$ be a convex constraint set, i.e. the solution shall be restricted to lie in $\mathcal{C}$. Thus, we want to solve

$$\underset{x \in \mathcal{V}}{\operatorname{argmin}} \{F(x)\} \text{ subject to } x \in \mathcal{C},$$

which could be e.g. a constrained least squares problem as in the first talk. This is equivalent so solving

$$\underset{x \in \mathcal{V}}{\operatorname{argmin}} \{F(x) + \delta_{\mathcal{C}}(x)\}.$$

When applying FISTA, $\delta_{\mathcal{C}}$ is the non-differentiable part taken care of via the prox operator, which equals the projection onto $\mathcal{C}$.

# Many other proximal gradient methods ...

- Alternating projection (projection onto convex sets)
- Douglas-Rachford splitting, alternating direction method of multipliers ADMM (popular e.g. in learning)
- Split-Bregman (basis pursuit problems, i.e. constrained $L^1$-minimization)
- Chambolle-Pock (swiss army knife if both $F$ and $G$ are non-differentiable, but convex with simple prox-operators).

All of these use the same basic building blocks as the ones shown in the previous section.

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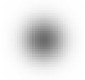
# Blurring

*The convolution $b * u$ with a kernel $b$ of total mass 1 can be interpreted as a blurring operation.*

Example: Gaussian blur (isotropic)

 *  = 

Example: Motion blur for diagonal motion (anisotropic)

 *  = 

# Total variation deblurring

**Inverse problem:** We observe the image $f$, which was created from an unknown original via

$$f = b * u + \text{ Gaussian noise.}$$

More general, $f = Au +$ noise (linear inverse problem).

**TV deblurring model**

Recover a deblurred image $u$ as the minimizer of

$$\underset{u}{\operatorname{argmin}} \left\{ \|\nabla u\|_1 + \frac{1}{2\lambda} \|b * u - f\|_2^2 \right\},$$

which is again the MAP estimate for Gaussian noise with TV prior.

# Example

*Original*

*Blurred + Noisy*

*Solution*

# FISTA for the general inverse problem

- The differentiable part is

$$F(u) = \frac{1}{2\lambda} \|Au - f\|_2^2 ,$$

which has functional derivative

$$dF(u) = \frac{1}{\lambda} A^T (Au - f),$$

which is Lipschitz-continuous with $L = \frac{1}{\lambda} \left\| A^T A \right\|_2$.

- The non-differentiable convex part is the regularizer, e.g. TV, with the proximation given by solving an instance of ROF.

# Inpainting problem

- Restoration problem: recover missing regions of an image
- Input:
  - damaged region $\Gamma \subset \Omega$
  - partial image $f : \Omega \setminus \Gamma \to \mathbb{R}$
- TV inpainting: Solve

$$\min_u \|\nabla u\|_1 \text{ such that } u = f \text{ on } \Omega \setminus \Gamma.$$

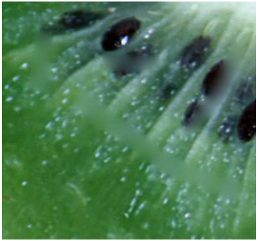

*Damaged image f*

*Recovered image u*

# TV inpainting results (“textured” images)

*Original*

*Damaged*

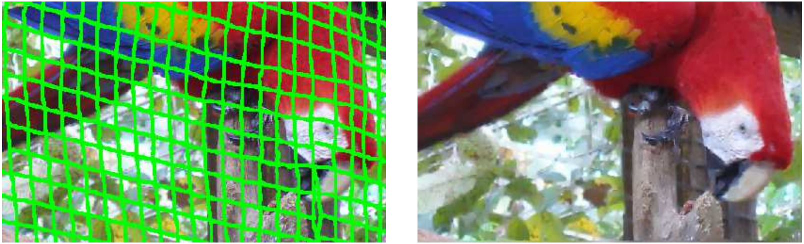


*Inpainted*

- TV inpainting is unconvincing for highly textured images if the missing regions are larger. The reason is that no structure is inferred from surrounding regions, and only boundary values of $\Gamma$ are taken into account.
- A better variational model for inpainting can be found e.g. in papers on **non-local TV** by Osher et al., or check methods based on **dictionary learning**.

# Object removal

Once we have an inpainting algorithm, we can employ it to remove unwanted regions in an image by marking them as damaged.



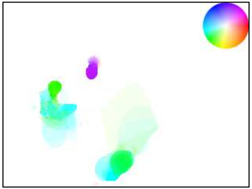
*Uncaging a bird*

# Optic flow

Input | Unknown

$I_t$ at time $t$

$I_{t+1}$ at time $t+1$



flow field $u : \Omega \to \mathbb{R}^2$

**TV-$\mathcal{L}^1$ optical flow (Zach et al. DAGM 2007)**

$$\operatorname*{argmin}_u \left\{ \underbrace{\mathrm{TV}(u_1) + \mathrm{TV}(u_2)}_{\text{regularizer } J(u)} + \frac{1}{2\lambda} \int_\Omega \underbrace{|I_{t+1}(x + u(x)) - I_t(x)|_1}_{\text{data term } \rho(u(x),x)} \, dx \right\}.$$

# Example: real-time optic flow

## Quadratic relaxation

**Idea: introduce auxiliary variable *v***

- Decouples regularizer from the data term.
- Let $\theta > 0$ be a coupling parameter, define family of energies in the variables $u, v$,

$$E_\theta(u, v) = \mathcal{J}(u) + \frac{1}{2\theta} \|u - v\|_2^2 + \int_\Omega \rho(x, v(x))\, dx.$$

- For $\theta \to 0$, the quadratic term forces $u$ to be close to $v$, so the solution of $E_\theta$ approximates the solution of $E$.

**Algorithm**

Set $\theta > 0$, start with initial $u_0, v_0$. Then iterate the following steps until convergence.

- Solve an instance of TV-$\mathcal{L}^2$:

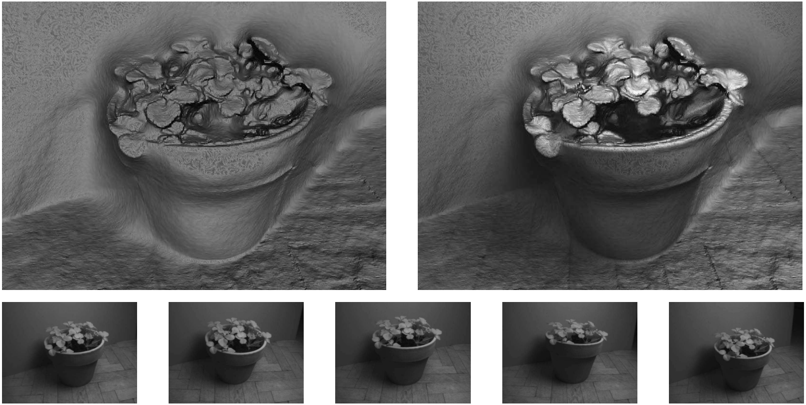
$$u_{k+1} = \underset{u}{\operatorname{argmin}} \, E_\theta(u, v_k).$$

- Solve the point-wise problem

$$v_{k+1} = \underset{v}{\operatorname{argmin}} \, E_\theta(u_{k+1}, v).$$

# Real-time depth

same technique (just reduced target dimension):
real-time disparity (depth) maps



Stuehmer et al. DAGM 2010

# Real-time scene flow

Wedel et al. ECCV 2008

# Overview

1. Introduction to variational methods
2. Convex optimization primer: proximal gradient methods
3. Variational inverse problems
4. **Segmentation and labeling problems**
5. 3D reconstruction
6. Summary

# The MAP estimate for binary segmentation

**Given:**

- An *observed image* $f$ on a domain $\Omega$,
  from which one computes (e.g. via a trained classifier)
- a *foreground probability distribution*

$$P_{fg}(x) = \text{Probability that the point } x \text{ is in the foreground}$$

- A *background probability distribution*

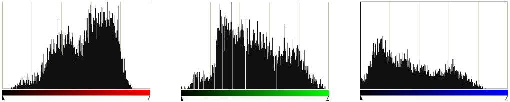
$$P_{bg}(x) = \text{Probability that the point } x \text{ is in the background}$$

**Wanted:**

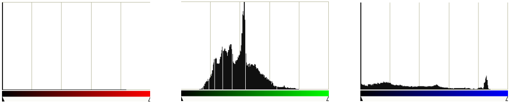
- The *MAP estimate for a binary segmentation* $u : \Omega \to \{0, 1\}$,
  where $u(x) = 1$ means a point is in the foreground, and $u(x) = 0$ means it is in the background:

$$\underset{u:\Omega\to\{0,1\}}{\operatorname{argmax}} \; P(u \mid f) = \underset{u:\Omega\to\{0,1\}}{\operatorname{argmax}} \; P(f \mid u)P(u).$$

# Example: histograms of user input (naive, illustration only)



Foreground histograms (RGB)



Background histograms (RGB)

$$P_{\text{fg}}(x) = \frac{\text{number of marked foreground pixels with color } f(x)}{\text{total number of marked foreground pixels}}$$

$$P_{\text{bg}}(x) = \frac{\text{number of marked background pixels with color } f(x)}{\text{total number of marked background pixels}}$$

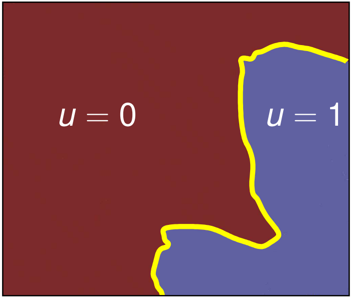
# MAP estimate for segmentation

Under the above assumptions, the MAP estimate for the segmentation is

$$\underset{u:\Omega\to\{0,1\}}{\operatorname{argmin}} \left\{ J(u) + \int_\Omega u \cdot \log\left[\frac{p_{bg}}{p_{fg}}\right] \mathrm{d}x \right\}.$$

Here, J(u) is the **prior.** A typical prior is the length of the interface.

# The binary case




Length of interface equals total variation of $u$,

$$\mathrm{TV}(u) = \int_\Omega |Du|$$

**Binary segmentation problem**

$$\underset{u:\Omega\to\{0,1\}}{\operatorname{argmin}}\ \mathrm{TV}(u) + \int_\Omega cu\,dx,$$

with local assigment costs $c$.

Space of binary functions $u : \Omega \to \{0, 1\}$ not convex
but: globally optimal solution possible by
relaxation to $u : \Omega \to [0, 1]$ and subsequent thresholding.

Chan, Esedoglu and Nikolova 2006

# FISTA for linear data terms

Straightforward:

- Since $F(u) = (c, u)$, the derivative is constant,

$$dF(u) = a.$$

- It is Lipschitz-continuous with arbitrary Lipschitz constant $L > 0$, one can e.g. just choose $L = 1$.

- Thus, the gradient descent step in $F$ just subtracts $c$ from the solution.

- Proximation in the TV-regularizer is an instance of ROF.

# Segmentation example

Input image

User marks

$\log(p_{bg}/p_{fg})$

Result

Simple model works pretty well with easy, clean input ...

# Segmentation example: noisy input

Input image

User marks

$\log(p_{bg}/p_{fg})$

Result

... but is not very robust against noise
(better classifiers requires, i.e. random forests)

Input with User marks  $\log(p_{bg}/p_{fg})$ (normalized)  TV (best result)

Length does not work well with elongated structures
(curvature regularity is better in that case, but complex)

Goldluecke and Cremers ICCV 2011

# Convex relaxation of the multilabel problem

**Indicator function** $u_\gamma : \Omega \to \{0,1\}$ assigned to each label $\gamma$:

$u_1 = 1$, all others zero

$u_2 = 1$, all others zero

$u_3 = 1$, all others zero

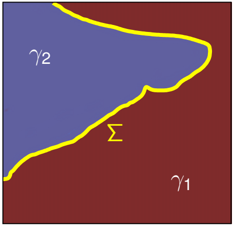
$u_4 = 1$, all others zero

$\sum_\gamma u_\gamma$ must be one !

**Problem relaxation**

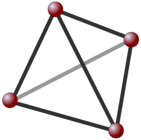
$$\operatorname*{argmin}_{u_\gamma:\Omega\to[0,1],\sum_\gamma u_\gamma=1} \left[ \mathcal{J}(\mathbf{u}) + \sum_\gamma \int_\Omega c_\gamma u_\gamma \, \mathrm{d}x \right],$$

# Regularization




The regularization penalty is proportional to the **label distance** times the **length of the interface**.

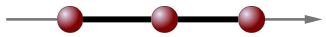
In this example $d(\gamma_1, \gamma_2) \cdot L(\Sigma)$



Euclidean representation of the label distance:

- Each label $\gamma$ is *represented* by a point $a_\gamma \in \mathbb{R}^k$.
- Label distance $d(\gamma, \mu) = |a_\gamma - a_\mu|_2$ .

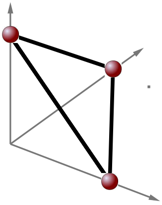
# Important special cases



$a_\gamma = \gamma \in \mathbb{R}$

**Ordered Labels**

- Example: depth reconstruction
- Can be solved globally with functional lifting [Pock, Schönemann, Graber, Bischof, Cremers '08]
- Continuous version of [Ishikawa '03]

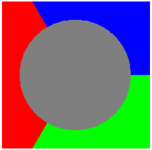


$a_\gamma = e_\gamma \in \mathbb{R}^N$

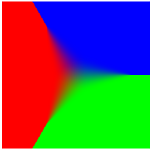
**Potts model**

- Example: segmentation
- No globally optimal solution possible if $N > 2$
- Continuous version of [Potts '52]

# Different Multilabel Regularizers



A comparison using the “triple-junction” problem

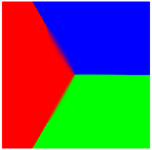


Zach, Gallup, Frahm, Niethammer ’08

$$\mathcal{J}_1(\mathbf{u}) = \frac{1}{2} \sum_{\gamma} \int_{\Omega} |Du_{\gamma}|$$

Lellmann, Kappes, Yuan, Becker, Schnörr ’08

$$\mathcal{J}_2(\mathbf{u}) = \int_{\Omega} \sqrt{\sum_{\gamma} \|Du_{\gamma}\|_A^2}, \quad \|v\|_A = \sqrt{v^T A^T A v}$$



Chambolle, Cremers, Pock ’08

$$\mathcal{J}_3(\mathbf{u}) = \int_{\Omega} \Psi(D\mathbf{u}), \Psi(\boldsymbol{q}) = \sup_{\boldsymbol{p}} \{\langle \boldsymbol{q}, \boldsymbol{p} \rangle \ : \ |p_{\gamma} - p_{\mu}| \leq d_{\gamma\mu}\}$$

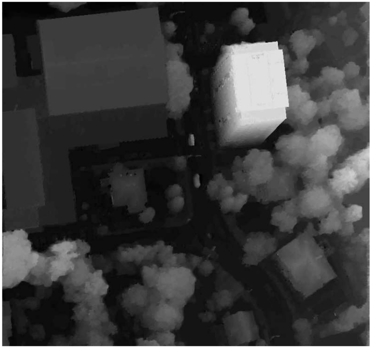
# Results: indoor stereo pair

- Stereo assignment cost $c_\gamma(x) = |I_{\text{left}}(x) - I_{\text{right}}(x + \gamma)|$
- Linear discontinuity penalty $\Rightarrow$ globally optimal solution.

*Images from UCSD lightfield repository*

# Results: aerial images (1400 x 1500 pixels)

One of two input images
(Courtesy of Microsoft Graz)



Depth reconstruction

# Examples: segmentation

Potts segmentation with $k = 10$ labels

Potts segmentation with $k = 16$ labels

# Ordering Constraints


*Labeling regions should have certain spatial relationships,*
*i.e. heaven is always above ground.*

## Direction-Aware New Regularizer

Strekalovskiy and Cremers, ICCV 2011

The labeling penalty may depend also on the **normal n of the interface** between two regions,

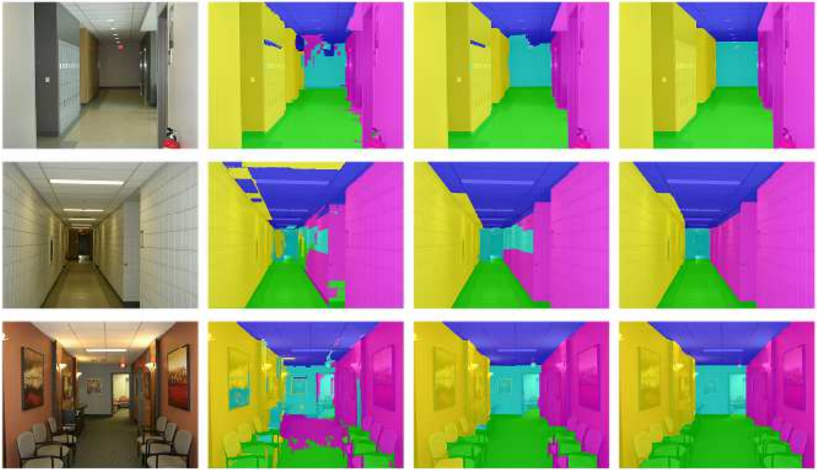
$$d : \Gamma \times \Gamma \times \mathbb{S} \to \mathbb{R}^+.$$

New direction-aware regularizer:

$$\mathcal{J}(\mathbf{u}) = \sup_{\mathbf{p} \in C} \sum_{\gamma} \int_{\Omega} \langle \mathbf{p}_\gamma, \nabla u_\gamma \rangle \, \mathrm{d}x,$$

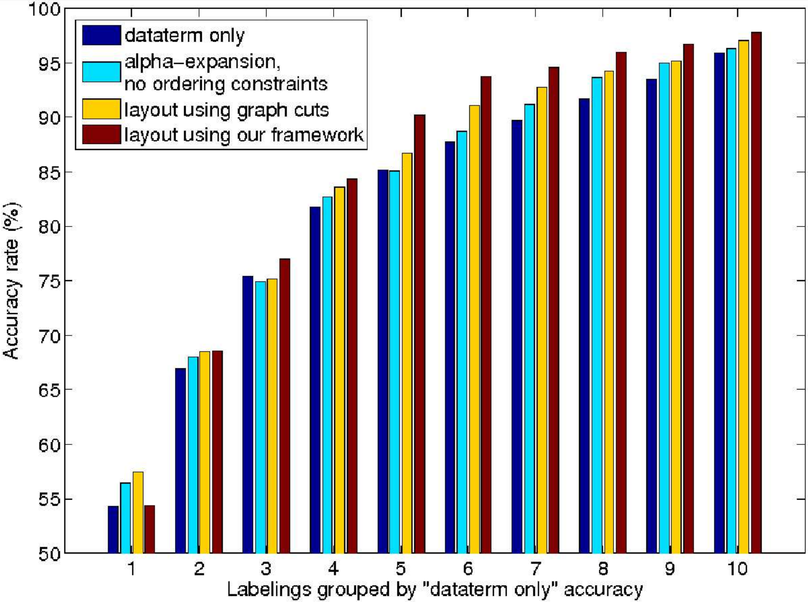
$$\text{with } C = \left\{ (\mathbf{p}_\gamma : \Omega \to \mathbb{R}^n) \;:\; \langle \mathbf{p}_\mu - \mathbf{p}_\gamma, \mathbf{n} \rangle \leq d(\gamma, \mu, \mathbf{n}) \quad \forall \gamma, \mu, \mathbf{n} \right\}.$$

# Results



Input Data term Potts Ordering

## Comparison


dataterm only
alpha–expansion, no ordering constraints
layout using graph cuts
layout using our framework
Accuracy rate (%)
Labelings grouped by "dataterm only" accuracy
100
95
90
85
80
75
70
65
60
55
50
1
2
3
4
5
6
7
8
9
10

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# The multiview reconstruction problem

**Given** $n$ images $\mathcal{I}_i : \Omega_i \to \mathbb{R}^3$
with projections $\pi_i : \mathbb{R}^3 \to \mathbb{R}^2$

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**Given** $n$ images $\mathcal{I}_i : \Omega_i \to \mathbb{R}^3$
with projections $\pi_i : \mathbb{R}^3 \to \mathbb{R}^2$

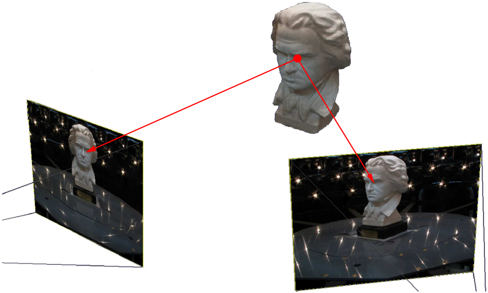
**Find** surface $\Sigma \subset \mathbb{R}^3$ with texture $T : \Sigma \to \mathbb{R}^3$ which optimally matches the input images.

# Variational 3D reconstruction

**Classical variational formulation (Faugeras and Keriven, 1998)**

Find a surface $\Sigma \subset \mathbb{R}^3$ which minimizes the photo-consistency error,

$$\operatorname*{argmin}_{\Sigma} \int_{\Sigma} \rho(s)\, ds.$$



- Point **on surface**
- projections look **similar** in different views
- **small value** of $\rho$

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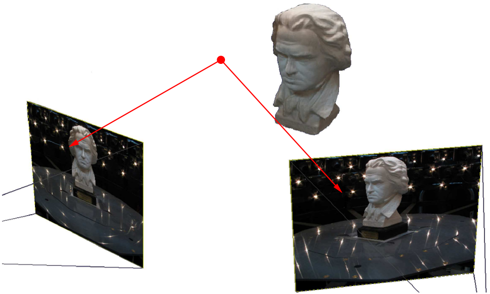


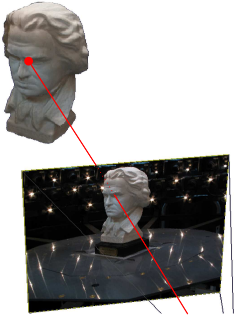
- Point **not on surface**
- projections look **different** in different views
- **large value** of $\rho$

# 3D reconstruction as a convex variational problem

**Convex functional** minimizes photo-consistency error:

$$\underset{u:\Omega\to\{0,1\}}{\operatorname{argmin}} \int_\Omega \rho\,|Du|$$

**Silhouette constraints** to avoid constant solutions



A ray through the **silhouette** must **intersect** the surface

Kolev, Klodt, Brox, Cremers IJCV'09

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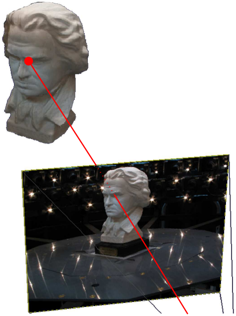


A ray through the **background** must **miss** the surface

Kolev, Klodt, Brox, Cremers IJCV'09

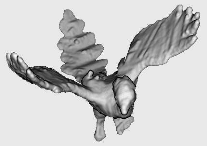
# Statues

Kolev and Matiouk, Akademisches Kunstmuseum Bonn 2010

# Superresolution texture maps

## Given

$n$ images $\mathcal{I}_i : \Omega_i \to \mathbb{R}$
projections $\pi_i : \mathbb{R}^3 \to \mathbb{R}^2$



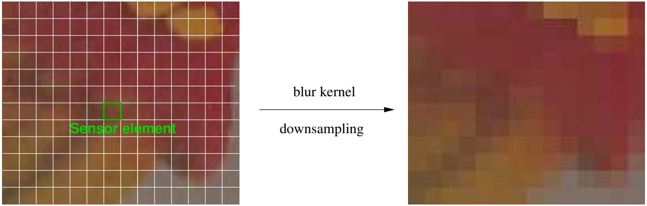
approximate surface $\Sigma$
(assumed Lambertian)

## Find

high-res color texture map
$T : \Sigma \to \mathbb{R}^3$
and accurate geometry

**Idea**:
Use information from multiple cameras for superresolution.

# Image formation in a sensor



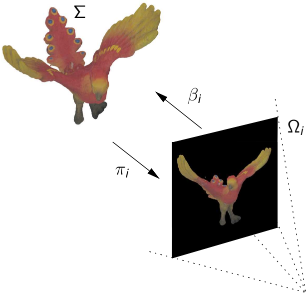

- Each sensor element samples incoming light over its area
- Sampling modeled by blur kernel $b$
- Leads to image formation model

$$\mathcal{I}_{\text{observed (low-res)}} = b * \mathcal{I}_{\text{incoming (high-res)}}$$

**Data term:** squared difference between input images and downsampled high-resolution rendering

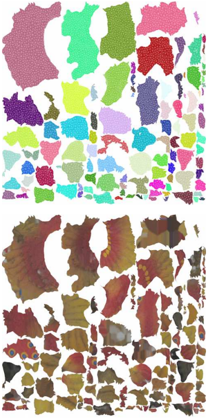
$$E(T) := \sum_{i=1}^{n} \int \Big( \underbrace{b * (T \circ \beta_i)}_{\text{Subsampled hi-res rendering}} - \underbrace{\mathcal{I}_i}_{\text{Input image}} \Big)^2 \, dx$$

**Regularizer:** total variation on the surface.

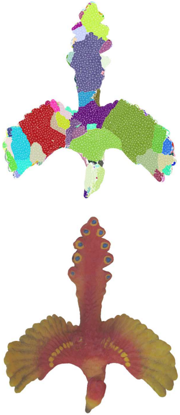



Goldluecke and Cremers, ICCV 2009

# Conformal texture atlas



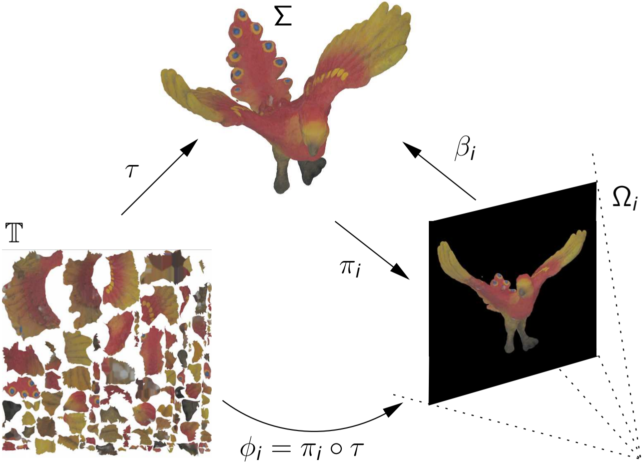
$$\mathbb{T} \xrightarrow{\mathcal{T}} \Sigma$$



Conformal atlas: transforms energy to 2D texture space

# Euler-Lagrange equation on texture space

$$\frac{1}{\lambda}\operatorname{div}\left(\sqrt{\lambda}\frac{\nabla\mathcal{T}}{\|\nabla\mathcal{T}\|}\right)+\sum_{i=1}^{n}\frac{v_i}{\sigma}\left((\mathcal{J}_i\mathcal{E}_i)\circ\phi_i\right)=0 \text{ on } \mathbb{T}$$



# Experiments

*Bird*

*Beethoven*

*Bunny*

- 30 cameras, input image resolution $768 \times 576$
- Initial geometry: Kolev and Cremers, ECCV 2008
- Texture resolution $2048 \times 2048$

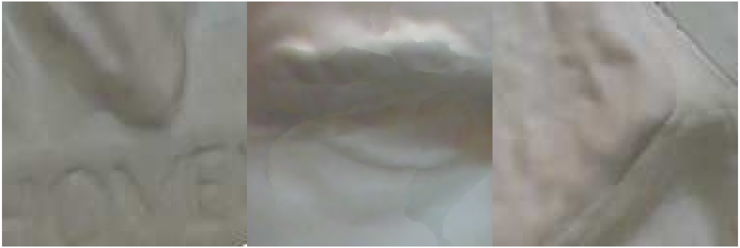
# Results: Bird

Rendered model

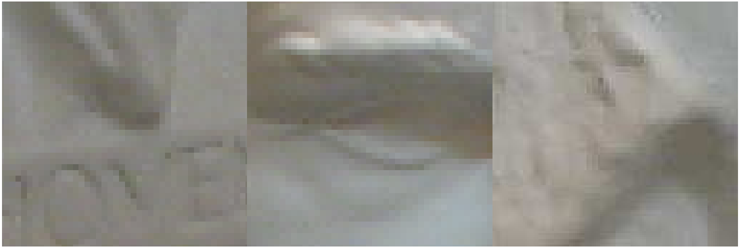
Input image

# Results: Beethoven

Rendered model



Input image



# Displacement optimization

Additional dependance on **displacement map** $D : \mathbb{T} \to \mathbb{R}$,

$$E(T, D) := \sum_{i=1}^{n} \int_{\hat{S}_i} \left( b * (T \circ \beta_i{}^{D}) - \mathcal{I}_i \right)^2 \mathrm{d}x + E_{\mathrm{tv}}(T, D).$$

- In $T$: energy is convex
- In $D$: multilabel problem with convex regularizer, **global optimization in $D$ possible**

Geometry

Estimated Texture

Input

Estimated displacement

Without displacement

With displacement

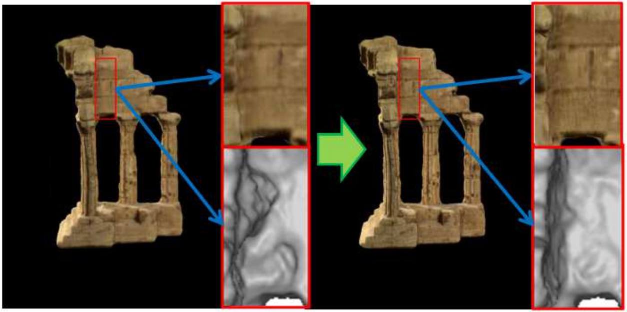
Goldlücke and Cremers DAGM 2009

# Variational camera calibration

**Idea:** assume the **projection parameters** $\pi$ **are unknowns** in the superresolution energy,

$$E(T,\pi) := \sum_{i=1}^{N} \int_{S_i} \|b * (T \circ \beta_i) - \mathcal{I}_i\| \, dx.$$

Optimize alternatingly for texture and projection. In a way, this can be thought of as a continuous version of bundle adjustment.



Goldluecke, Aubry, Kolev, Cremers, IJCV 2013.

# Textured bunny model

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## Summary

- Variational methods: a powerful tool for realistic modeling
- Non-convex energies: gradient descent, local minimum, usually requires good initialization
- Convex energies: no local minima, powerful optimization tools (e.g. proximal gradient methods)
- For $\text{argmin}_u\{F(u) + G(u)\}$ with convex $G$ and differentiable and convex $F$, FISTA is a good general-purpose method with optimal convergence rate.
- If $F$ is only convex, check out e.g. [Chambolle-Pock 2012] or [Nesterov 2005].
- Numerous appliations: variational inverse problems, segmentation and labeling, 3D reconstruction ...